

The Dirac-Nambu-Goto p -Branes as Particular Solutions to a Generalized, Unconstrained Theory¹

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ABSTRACT

The theory of the usual, constrained p -branes is embedded into a larger theory in which there is no constraints. In the latter theory the Fock- Schwinger proper time formalism is extended from point-particles to membranes of arbitrary dimension. For this purpose the tensor calculus in the infinite dimensional membrane space \mathcal{M} is developed and an action which is covariant under reparametrizations in \mathcal{M} is proposed. The canonical and Hamiltonian formalism is elaborated in detail. The quantization appears to be straightforward and elegant. No problem with unitarity arises. The conventional p -brane states are particular stationary solutions to the functional Schrödinger equation which describes the evolution of a membrane's state with respect to the invariant evolution parameter τ . A τ dependent solution which corresponds to the wave packet of a null p -brane is found. It is also shown that states of a lower dimensional membrane can be considered as particular states of a higher dimensional membrane.

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1. Introduction

The theory of relativistic membranes of arbitrary dimension (p -branes) which include point particles ($p = 0$) and strings ($p = 1$) is very elegant and is explored in great detail [1],[2]. Many interesting results are obtained, including the possibility of unifying the fundamental interactions. In view of such importance of the relativistic p -branes it is desirable to obtain a broader look at the theory and find some interrelation not known so far. Such a goal is attempted in the present paper.

A motivation is to use, instead of the Dirac-Nambu-Goto action which is practically unquantizable for a generic p -brane, another action which is quantizable. There exists a very elegant approach by Schild [3] in which the string action contains the first power of the determinant g of the worldsheet metric (and not the square root). The equations of motion are those of a Nambu-Goto string and at the same time the gauge is automatically fixed so that g is constant. The string analog of the Hamilton-Jacobi formalism was elaborated by Eguchi [4] who found that the world sheet area has the role of evolution parameter. The Schild action is a straightforward generalization of the well known point-particle action [5] $\frac{1}{2} \int d\tau (\frac{dx^\mu}{d\tau})^2$ which I shall call here the Stueckelberg action.

In recent years I have been exploring another possible generalization of the Stueckelberg action to p -branes [6]-[9]. The generalized p -brane action looks essentially like the Stueckelberg action, except for the integration over the additional parameters ξ^a due to the extension of the object, and contains the invariant evolution parameter τ and the background fields Λ , Λ^a . It is *covariant* with respect to reparametrizations of τ , but it is not *invariant* and thus there is no constraints. In Ref. [9] the equations of motion for the case $\Lambda^a = 0$ were derived and the theory was quantized. The aim of the present paper is to consider the more general case $\Lambda^a \neq 0$. A very interesting relation involving expressions for p -brane constraints (Eq. 37) is derived. The canonical and Hamiltonian formalism is elaborated in detail. Then we quantize the theory and show that particular stationary states (with respect to τ) correspond to states of the conventional, constrained p -brane theory. We also obtain that the wave functional of a p -dimensional membrane can represent, in a limiting case, a $(p - 1)$ -dimensional membrane. Since this process can be continued, it holds that states of a lower dimensional membrane are contained among the states of a higher dimensional membrane. Finally, we consider a simple solution which corresponds to the case of a null membrane discussed by Schild [3] and Roshchupkin et al. [10].)

2. The space of unconstrained membranes

The basic objects of the theory we are going to discuss in this paper are n -dimensional, arbitrarily deformable and hence unconstrained, membranes \mathcal{V}_n living in an N -dimensional space V_N . The dimensions n and N , as well as the corresponding signatures are left unspecified at this stage. An unconstrained membrane \mathcal{V}_n is represented by the embedding functions $X^\mu(\xi^a)$, $\mu = 0, 1, 2, \dots, N - 1$, where ξ^a , $a = 0, 1, 2, \dots, n - 1$, are local parameters (coordinates) on \mathcal{V}_n . The set of all possible membranes \mathcal{V}_n , with n fixed, forms an infinite dimensional space \mathcal{M} . A membrane \mathcal{V}_n can be considered as a point in \mathcal{M} parametrized

by coordinates $X^\mu(\xi^a) \equiv X^{\mu(\xi)}$ which bear a discrete index μ and n continuous indices ξ^a . To the discrete index μ we can ascribe arbitrary numbers: instead of $\mu = 0, 1, 2, \dots, N-1$ we may set $\mu' = 1, 2, \dots, N$ or $\mu' = 2, 5, 3, 1, \dots$, etc. In general,

$$\mu' = f(\mu) \quad (1)$$

where f is a transformation. Analogously, a continuous index ξ^a can be given arbitrary continuous values. Instead of ξ^a we may take ξ'^a which are functions of ξ^a :

$$\xi'^a = f^a(\xi) \quad (2)$$

As far as we consider, respectively, μ and ξ^a as a discrete and a continuous index of coordinates $X^{\mu(\xi)}$ in the infinite dimensional space \mathcal{M} , reparametrization of ξ^a is analogous to renumbering of μ . Both kinds of transformations, (1) and (2), refer to the same point of the space \mathcal{M} ; they are *passive transformations*. For instance, under the action of (2) we have

$$X'^{\mu}(\xi') = X'^{\mu}(f(\xi)) = X^{\mu}(\xi) \quad (3)$$

which says that the same point \mathcal{V}_n can be described either by functions $X^\mu(\xi)$ or $X'^{\mu}(\xi)$ (where we may write $X'^{\mu}(\xi)$ instead of $X'^{\mu}(\xi')$ since ξ' is a running parameter and can be renamed into ξ).

Then there exist also *the active transformations* which transform one point of the space \mathcal{M} into another. Given a parametrization of ξ^a and a numbering of μ , a point \mathcal{V}_n of \mathcal{M} with coordinates $X^\mu(\xi)$ can be transformed into another point \mathcal{V}'_n with coordinates $X'^{\mu}(\xi)$. Parameters ξ^a are now considered as "body fixed", so that distinct functions $X^\mu(\xi)$, $X'^{\mu}(\xi)$ represent distinct points \mathcal{V}_n , \mathcal{V}'_n of \mathcal{M} . Physically, these are distinct membranes which may be deformed one with respect to the other. Such a membrane is *unconstrained*, since all coordinates $X^\mu(\xi)$ are necessary for its description [7]-[9]. In order to distinguish an unconstrained membrane \mathcal{V}_n from the corresponding mathematical manifold V_n , we use different symbols \mathcal{V}_n and V_n .

It may happen, in particular, that two distinct membranes \mathcal{V}_n and \mathcal{V}'_n both lie on the same mathematical surface V_n , and yet they are physically distinct objects, represented by different points in \mathcal{M} .

The concept of an unconstrained membrane can be illustrated by imaging a rubber sheet spanning a surface V_2 . The sheet can be deformed from one configuration (let me call it \mathcal{V}_2) into another configuration \mathcal{V}'_2 in such a way that both configurations \mathcal{V}_2 , \mathcal{V}'_2 are spanning the same surface V_2 . The configurations \mathcal{V}_2 , \mathcal{V}'_2 are described by functions $X^i(\xi^1, \xi^2)$, $X'^i(\xi^1, \xi^2)$ ($i = 1, 2, 3$), respectively. The latter functions, from the mathematical point of view, both represent the same surface V_2 , and can be transformed one into the other by a reparametrization of ξ^1, ξ^2 . But from the physical point of view, $X^i(\xi^1, \xi^2)$ and $X'^i(\xi^1, \xi^2)$ represent two different configurations of the rubber sheet.

The reasoning presented in the last few paragraphs implies that, since our membranes are assumed to be arbitrarily deformable, different functions $X^\mu(\xi)$ can always represent physically different membranes. This justifies use of the coordinates $X^\mu(\xi)$ for the description of points in \mathcal{M} . Later, when we shall consider membrane's dynamics, we shall admit τ -dependence of coordinates $X^\mu(\xi)$. In this section, all expressions refer to a fixed value of τ , therefore we omit it in the notation.

In analogy to the finite dimensional case we can introduce the distance $d\ell$ in our infinite dimensional space \mathcal{M} :

$$d\ell^2 = \int d\xi d\zeta \rho_{\mu\nu}(\xi, \zeta) dX^\mu(\xi) dX^\nu(\zeta) = \rho_{\mu(\xi)\nu(\zeta)} dX^{\mu(\xi)} dX^{\nu(\zeta)} = dX^{\mu(\xi)} dX_{\mu(\xi)} \quad (4)$$

where $\rho_{\mu\nu}(\xi, \zeta) = \rho_{\mu(\xi)\nu(\zeta)}$ is the metric in \mathcal{M} . In eq.(4) we use a notation, similar to one that is usually used when dealing with more evolved functional expressions [11], [12]. In order to distinguish continuous indices from the discrete ones, the former are written within brackets. When we write $\mu(\xi)$ as a subscript or superscript this denotes a pair of indices μ and (ξ) (and not that μ is a function of ξ). We also use the convention that summation is performed over repeated indices (such as a, b) and integration over repeated continuous indices (such as $(\xi), (\zeta)$).

The tensor calculus in \mathcal{M} [9] is analogous to one in a finite dimensional space. The differential of coordinates $dX^\mu(\xi) = dX^{\mu(\xi)}$ is a vector in \mathcal{M} . The coordinates $X^{\mu(\xi)}$ can be transformed into new coordinates $X'^{\mu(\xi)}$ which are functionals of $X^{\mu(\xi)}$:

$$X'^{\mu(\xi)} = F^{\mu(\xi)}[X] \quad (5)$$

The transformation (5) is very important. It says that if functions $X^\mu(\xi)$ represent a membrane \mathcal{V}_n then any other functions $X'^\mu(\xi)$ obtained from $X^\mu(\xi)$ by a functional transformation also represent the same membrane \mathcal{V}_n . In particular, under reparametrization of ξ^a the functions $X^\mu(\xi)$ change into new functions; a reparametrization thus manifests itself as a special functional transformation which belongs to a subclass of the general functional transformations (5).

Under a general coordinate transformation (5) a generic vector $A^{\mu(\xi)} \equiv A^\mu(\xi)$ transforms as ³

$$A^{\mu(\xi)} = \frac{\partial X'^{\mu(\xi)}}{\partial X^{\nu(\zeta)}} A^{\nu(\zeta)} \equiv \int d\zeta \frac{\delta X'^{\mu(\xi)}}{\delta X^{\nu(\zeta)}} A^{\nu(\zeta)} \quad (6)$$

Similar transformations hold for a covariant vector $A_{\mu(\xi)}$, a tensor $B_{\mu(\xi)\nu(\zeta)}$, etc. Indices are lowered and raised, respectively by $\rho_{\mu(\xi)\nu(\zeta)}$ and $\rho^{\mu(\xi)\nu(\zeta)}$, the latter being the inverse metric tensor satisfying

$$\rho^{\mu(\xi)\alpha(\eta)} \rho_{\alpha(\eta)\nu(\zeta)} = \delta^{\mu(\xi)}_{\nu(\zeta)} \quad (7)$$

Suitable choice of the metric - assuring the invariance of the line element (4) under the transformations (2) and (5) - is for instance

$$\rho_{\mu(\xi)\nu(\zeta)} = \sqrt{|f|} \alpha g_{\mu\nu} \delta(\xi - \zeta) \quad (8)$$

where $f \equiv \det f_{ab}$ is the determinant of the induced metric $f_{ab} \equiv \partial_a X^\alpha \partial_b X^\beta g_{\alpha\beta}$ on the sheet V_n , $g_{\mu\nu}$ is the metric tensor of the embedding space V_N and α an arbitrary function of ξ^a .

With the metric (8) the line element (4) becomes

$$d\ell^2 = \int d\xi \sqrt{|f|} \alpha g_{\mu\nu} dX^{\mu(\xi)} dX^{\nu(\xi)} \quad (9)$$

³ A similar formalism, but for a specific type of the functional transformations (5), namely the reparametrizations which functionally depend on string coordinates, was developed by Bardakci [11]

Rewriting the abstract formulas back into the usual notation with explicit integration we have

$$A^{\mu(\xi)} = A^\mu(\xi) \quad (10)$$

$$A_{\mu(\xi)} = \rho_{\mu(\xi)\nu(\zeta)} A^{\nu(\zeta)} = \int d\zeta \rho_{\mu\nu}(\xi, \zeta) A^\nu(\xi) = \sqrt{|f|} \alpha g_{\mu\nu} A^\nu(\xi) \quad (11)$$

The inverse metric is

$$\rho^{\mu(\xi)\nu(\zeta)} = \frac{1}{\alpha \sqrt{|f|}} g^{\mu\nu} \delta(\xi - \zeta) \quad (12)$$

Indeed, from (7), (8) and (12) we obtain

$$\delta^{\mu(\xi)}_{\nu(\zeta)} = \int d\eta g^{\mu\sigma} g_{\nu\sigma} \delta(\xi - \eta) \delta(\zeta - \eta) = \delta^\mu_\nu \delta(\xi - \zeta) \quad (13)$$

The invariant volume element (measure) of our membrane space \mathcal{M} is [13]

$$\mathcal{D}X = (\text{Det } \rho_{\mu\nu}(\xi, \zeta))^{1/2} \prod_{\xi, \mu} dX^\mu(\xi) \quad (14)$$

Here Det denotes a continuum determinant taken over ξ, ζ as well as over μ, ν . In the case of the diagonal metric (8) we have

$$\mathcal{D}X = \prod_{\xi, \mu} \left(\sqrt{|f|} \alpha |g| \right)^{1/2} dX^\mu(\xi) \quad (15)$$

As in a finite dimensional space we can now define covariant derivative also in \mathcal{M} . For a scalar functional $A[X(\xi)]$ the covariant functional derivative coincides with the ordinary functional derivative:

$$A_{;\mu(\xi)} = \frac{\delta A}{\delta X^\mu(\xi)} \equiv A_{,\mu(\xi)} \quad (16)$$

But in general a geometric object in \mathcal{M} is a tensor of arbitrary rank, $A^{\mu_1(\xi_1)\mu_2(\xi_2)\dots}_{\nu_1(\zeta_1)\nu_2(\zeta_2)\dots}$, which is a functional of $X^\mu(\xi)$, and its covariant derivative contains the affinity $\Gamma^{\mu(\xi)}_{\nu(\zeta)\sigma(\eta)}$ composed of the metric (8) [9]. For instance, for a vector we have

$$A^{\mu(\xi)}_{;\nu(\zeta)} = A^{\mu(\xi)}_{,\nu(\zeta)} + \Gamma^{\mu(\xi)}_{\nu(\zeta)\sigma(\eta)} A^{\sigma(\eta)} \quad (17)$$

Let the alternative notations for ordinary and covariant functional derivative be analogous to ones used in a finite dimensional space:

$$\frac{\delta}{\delta X^\mu(\xi)} \equiv \frac{\partial}{\partial X^\mu(\xi)} \equiv \partial_{\mu(\xi)} \quad , \quad \frac{D}{DX^\mu(\xi)} \equiv \frac{D}{DX^\mu(\xi)} \equiv D_{\mu(\xi)} \quad (18)$$

3. The classical theory of unconstrained membranes

3.1. The dynamics of unconstrained membranes

Now we are going to describe motion of a membrane \mathcal{V}_n in an embedding space V_N . For this purpose we introduce an extra parameter, the evolution time τ , and assume that the embedding functions depend also on τ , so that the parametric equation of a moving membrane is

$$X^\mu = X^\mu(\tau, \xi^a) \quad (19)$$

We assume that the action is given by the following expression

$$I = \frac{\kappa}{2} \int d\tau d^n\xi \sqrt{|f|} \left(\frac{1}{\Lambda} g_{\mu\nu} (\dot{X}^\mu + \Lambda^a \partial_a X^\mu) (\dot{X}^\nu + \Lambda^b \partial_b X^\nu) + \Lambda \right) \quad (20)$$

Here κ is a constant, and dot denotes the derivative with respect to τ . If we treat Λ and Λ^a as the quantities to be varied, i.e. the Lagrange multipliers, then one immediately finds that they give the well known p -brane constraints [14] and that the action (20) is equivalent [8] to the Dirac-Nambu-Goto action. In Refs. [6]-[9], on the contrary, we proposed to treat Λ and Λ^a as the background, τ -dependent fields defined over membrane's coordinates ξ^a . Eq.(20) can then be considered as an approximation to a full action with a kinetic term for Λ and Λ^a . The latter fields are presumably dynamical fields for a system of large number of membranes, but are effectively the background fields for a single member of the system. Though it would be very interesting to investigate possible full actions and mechanisms leading to (20) we prefer first to explore the consequences of the action (20). And if they manifest themselves sufficiently interesting, this will justify a search for a more fundamental principle behind the action (20).

In a given parametrization, Λ and Λ^a are fixed, prescribed functions of τ and ξ^a . Under a reparametrization $\xi^a \rightarrow \xi'^a$ the quantities Λ and Λ^a are assumed to transform, respectively, as a scalar and a vector in the n -dimensional space V_n associated with the membrane \mathcal{V}_n , and under reparametrizations of τ they are assumed to transform according to $\Lambda' = (d\tau/d\tau')\Lambda$ and $\Lambda'^a = (d\tau/d\tau')\Lambda^a$. The action (20) then does not change its form; it is given by the same expression (20) in which unprimed quantities are replaced by primed ones:

$$I'[X'^\mu \Lambda', \Lambda'^a] = I[X'^\mu \Lambda', \Lambda'^a] = I[X^\mu \Lambda, \Lambda^a] \quad (21)$$

Though the action (20) is covariant with respect to reparametrizations of ξ^a and τ it is not invariant, since the transformed action contains the transformed functions Λ' , Λ'^a . The latter are not the quantities to be varied (not Lagrange multipliers) and, consequently, the transformed action $I'[X'^\mu]$ is not the same functional of the dynamical variables as is the original action $I[X^\mu]$.

In the action (20) the dimensions n and N , and the signatures of the corresponding manifolds V_n and V_N are left unspecified. So the action (20) contains many possible particular cases. Especially interesting are the following ones:

Case 1. The manifold V_n belonging to an unconstrained membrane \mathcal{V}_n has the signature $(+ - - \dots)$ and corresponds to an n -dimensional *worldsheet* with one time-like and $n - 1$ space-like dimensions. The index of the worldsheet coordinates assumes the values $a = 0, 1, 2, \dots, n - 1$.

Case 2. The manifold V_n belonging to our membrane \mathcal{V}_n has the signature (- - - ...) and corresponds to a space-like p -brane; therefore we take $n = p$. The index of the membrane's coordinates ξ^a assumes the values $a = 1, 2, \dots, p$.

Throughout the paper we shall use the single formalism and apply it, when convenient, either to the *Case 1* or to the *Case 2*.

When the dimension of the manifold V_n belonging to \mathcal{V}_n is $n = p + 1$ and the signature is (+ - - - ...), i.e. when we consider *Case 1*, then the action (20) reduces to the action of the usual Dirac-Nambu-Goto p -dimensional membrane (shortly p -brane) as follows. Putting $\Lambda^a = 0$ and $\dot{X}^\mu = 0$ Eq.(20) becomes

$$I = \frac{\kappa}{2} \int \Lambda d\tau d^n \xi \sqrt{|f|} \quad (22)$$

which is equal, when Λ is independent of ξ^a , to the minimal surface action, apart from the integration over $d\tau$, which brings an extra constant factor $\frac{\kappa}{2} \int \Lambda d\tau$.

More generally, when $\Lambda^a \neq 0$, we obtain the action (22) if in Eq.(20) we put $\dot{X}^\mu + \Lambda^a \partial_a X^\mu = 0$. Then the N -velocity \dot{X}^μ is tangent to the surface $X^\mu(\xi)$, and the n -velocity in V_n is equal to $-\Lambda^a$.

An advantage of considering a p -brane's worldsheet as a particular solution to the equations of motion derived from (20) is in the fact that the action (20) implies no dynamical constraints among the variables X^μ and the corresponding canonical momenta $p_\mu = (\kappa \sqrt{|f|}/\Lambda)(\dot{X}_\mu + \Lambda^a \partial_a X_\mu)$.

A question now arises about a physical meaning of the parameter τ [5],[6]-[9]. In the case of a point particle a parametrization of τ can be chosen such that Λ is a constant. Then τ is proportional to the proper time, i.e. to length of the worldline, and by suitably choosing the constant of motion, τ becomes equal to proper time.

In a generic case of an n -dimensional membrane the action (20) can be written in a compact form by using the tensor notation of \mathcal{M} space (Sec.2.) and taking $\alpha = \kappa/\Lambda$ in the metric (8):

$$I = \frac{1}{2} \int d\tau \left(\dot{X}^{\mu(\xi)} \dot{X}_{\mu(\xi)} + K \right) \quad (23)$$

where

$$K \equiv \int d^n \xi \sqrt{|f|} \Lambda \kappa \quad (24)$$

and where, for simplicity, we consider for the moment the case $\Lambda^a = 0$ ⁴. The action (23) describes dynamics of a "point particle" in \mathcal{M} . It is invariant under (i) renumbering of μ (Eq.(1)), (ii) reparametrizations of ξ^a (Eq.(2)), (iii) general coordinate transformations in the embedding spacetime V_N (including Lorentz transformations, when V_N is flat), and (iv) the general coordinate transformations (5) in \mathcal{M} (including the transformations (ii) and (iii) as special cases). When Λ is a constant in τ , then the quantity $\dot{X}^{\mu(\xi)} \dot{X}_{\mu(\xi)} - K$ is a constant of motion, C . So we have $dX^{\mu(\xi)} dX_{\mu(\xi)} = (K + C) d\tau^2$, i.e., the line element of the trajectory in \mathcal{M} ("proper time") is proportional to τ . We can choose parametrization

⁴Such a choice is especially distinctive, because the action (20) then becomes invariant (not only covariant) under reparametrizations of membrane's coordinates ξ^a . Since the action is still not invariant under reparametrizations of the evolution parameter τ , there is no dynamical constraints, and all the variables $X^\mu(\xi)$ and the momenta $p_\mu(\xi)$ remain independent (see also refs. [6]-[9]).

of τ and/or the constant of motion C such that $K + C = 1$; then τ is just equal to proper time in \mathcal{M} . Simplicity of the action (23) and the direct relation between τ and the proper time in \mathcal{M} justifies our choice to consider this alternative unconstrained p -brane action, instead of the Schild action [3].

In flat embedding spacetime ($g_{\mu\nu} = \eta_{\mu\nu}$) the equations of motion, derived from the action (20) are

$$\frac{1}{\sqrt{|f|}} \frac{d}{d\tau} \left(\sqrt{|f|} \partial X_\mu \frac{1}{\Lambda} \right) + \frac{1}{\sqrt{|f|}} \partial_a \left(\sqrt{|f|} \left(\partial X_\mu \frac{\Lambda^a}{\Lambda} + \partial^a X_\mu \mu \right) \right) = 0 \quad (25)$$

where

$$\partial X^\mu \equiv \dot{X}^\mu + \Lambda^a \partial_a X^\mu \quad (26)$$

$$\mu \equiv \frac{1}{2} \left(\frac{\partial X_\mu \partial X^\mu}{\Lambda} + \Lambda \right) \quad (27)$$

Let us first observe that $\frac{1}{\sqrt{|f|}} \partial_a (\sqrt{|f|} \partial^a X_\mu) = D_a D^a X_\mu$, where D_a denotes the covariant derivative with respect to coordinates ξ^a . The induced metric on V_n is $\gamma_{ab} = \partial_a X^\mu \partial_b X_\mu$. Since the covariant derivative of γ_{ab} is zero, we have the following identity

$$\partial_c X^\mu D_a D_b X_\mu = 0 \quad (28)$$

Contracting (25) by $\partial^c X^\mu$ and using (28) we find

$$\partial_c \mu = - \frac{1}{\sqrt{|f|}} \frac{d}{d\tau} \left(\sqrt{|f|} \partial X_\mu \frac{1}{\Lambda} \right) \partial^c X^\mu - \frac{1}{\sqrt{|f|}} \partial_a \left(\sqrt{|f|} \partial X_\mu \frac{\Lambda^a}{\Lambda} \right) \partial^c X^\mu \quad (29)$$

Inserting (29) into the original equations of motion (25) we obtain

$$b^\nu{}_\mu \left[\frac{1}{\sqrt{|f|}} \frac{d}{d\tau} \left(\sqrt{|f|} \partial X_\nu \frac{1}{\Lambda} \right) + \frac{1}{\sqrt{|f|}} \partial_a \left(\sqrt{|f|} \partial X_\nu \frac{\Lambda^a}{\Lambda} \right) \right] + \mu D_a D^a X_\mu = 0 \quad (30)$$

where

$$b^\nu{}_\mu \equiv \delta^\nu{}_\mu - \partial_a X^\nu \partial^a X_\mu \quad (31)$$

From Eq.(30) it follows that the equation of minimal surface

$$D_a D^a X_\mu = 0 \quad (32)$$

is satisfied, provided that

$$b^\nu{}_\mu \left[\frac{dp_\nu}{d\tau} + \partial_a (p_\nu \Lambda^a) \right] = 0 \quad (33)$$

Here we have denoted

$$p_\nu = \frac{\kappa \sqrt{|f|} \partial X_\nu}{\Lambda} \quad (34)$$

where $p_\nu = \partial \mathcal{L} / \partial \dot{X}^\nu$ is the canonical momentum density belonging to the action (20).

In particular, Eq.(33) is satisfied when

$$\partial X_\mu = 0 \quad (35)$$

This is just the case we had in deriving the action (22) from (20). However, we must take into account also Eq.(29). We see that the latter equation can be satisfied together with (35) only when

$$\partial_a \Lambda = 0 \quad (36)$$

This is consistent with the fact that the action (22) describes a minimal surface if Λ is independent of ξ^a .

In general, when $\partial_a \Lambda \neq 0$, from Eqs.(29),(34) we have

$$\frac{d}{d\tau}(p_\mu \partial_c X^\mu) + \partial_a(\Lambda^a p_\mu \partial_c X^\mu) = \frac{1}{2} \partial_c \Lambda \frac{\sqrt{|f|}}{\kappa} \left(\frac{p^2}{|f|} - \kappa^2 \right) \quad (37)$$

If a particular solution $X^\mu(\tau, \xi^a)$ to the membrane's equations of motion satisfies

$$p_\mu \partial_c X^\mu = 0 \quad (38)$$

then, because of (37), it automatically satisfies

$$p^2 - |f| \kappa^2 = 0 \quad (39)$$

The latter equations (38),(39) are the well known constraints for an n -dimensional membrane [14]. In particular, if we consider *Case 2* they are just constraints for a usual p -brane, and a solution $X^\mu(\tau, \xi)$ satisfying (38) then describes a minimal $p+1$ dimensional surface.

In general, for our unconstrained membrane

$$p_\mu \partial_a X^\mu \neq 0 \quad (40)$$

and $X^\mu(\tau, \xi)$ does not describe a minimal surface. The momentum p_μ has a nonvanishing tangent component, which means that in general there is the intrinsic motion within the membrane, i.e. different parts of the membrane move relative to each other. Such unconstrained membranes are closely related to the wiggly membranes [15],[7],[8].

In particular, when Eq.(38) is satisfied, the momentum p_μ is perpendicular (in Minkowski space V_N) to the membrane X^μ . We can distinguish two cases:

(i) $\Lambda^a = 0$; then Eq.(38) becomes $\partial X^\mu \partial_a X_\mu = \dot{X}^\mu \partial_a X_\mu = 0$ which means that there is no intrinsic motion within the membrane (no wiggleness).

(ii) $\Lambda^a \neq 0$; then Eq.(38) becomes $\partial X^\mu \partial_a X_\mu = (\dot{X}^\mu + \Lambda_c \partial^c X^\mu) \partial_a X_\mu = \dot{X}^\mu \partial_a X_\mu + \Lambda_a = 0$ which implies that there is an intrinsic motion within the membrane. Our membrane, though sweeping a minimal surface, is thus a wiggly membrane and wiggleness is determined by Λ^a .

2.3 The canonical formalism and the Hamiltonian

The action (20) implies no constraints, therefore the canonical and Hamiltonian formalism [16] is straightforward. The canonical momentum density $p_\mu(\tau, \xi)$ satisfying the following equal τ Poisson bracket relations

$$\{X^\mu(\xi), p_\nu(\xi')\} = \delta^\mu_\nu \delta(\xi - \xi') \quad (41)$$

$$\{X^\mu(\xi), X^\nu(\xi')\} = 0, \quad \{p_\mu(\xi), p_\nu(\xi')\} = 0 \quad (42)$$

The Poisson bracket of two generic functionals $A[X^\mu(\xi), p_\nu(\xi)]$ and $B[X^\mu(\xi), p_\nu(\xi)]$ is defined by

$$\{A, B\} = \int d^n \xi'' \left(\frac{\delta A}{X^\alpha(\xi'')} \frac{\delta B}{\delta p_\alpha(\xi'')} - \frac{\delta B}{X^\alpha(\xi'')} \frac{\delta A}{\delta p_\alpha(\xi'')} \right) \quad (43)$$

Now we are going to derive the Hamiltonian belonging to the action (20). For this purpose we vary both the field quantities $X^\mu(\tau, \xi)$ and the boundary B of the integration region R . In general, for an action with a nonsingular Lagrangian density $\mathcal{L}(X^\mu, \dot{X}^\mu, \partial_a X^\mu)$ (which implies no constraints)

$$I = \int d\tau d^n \xi \mathcal{L}(X^\mu, \dot{X}^\mu, \partial_a X^\mu) \quad (44)$$

we have

$$\begin{aligned} \bar{\delta} I &= \int_R d\tau d^n \xi \delta \mathcal{L} + \int_{R-R'} d\tau d^n \xi \mathcal{L} = \int_R d\tau d^n \xi \delta \mathcal{L} + \int d\tau \int_B d\Sigma_a \mathcal{L} \delta \xi^a + \int d^n \xi \mathcal{L} \delta \tau \Big|_{\tau_1}^{\tau_2} \\ &= \int d\tau d^n \xi \left[\delta \mathcal{L} + \partial_a (\mathcal{L} \delta \xi^a) + \frac{\partial}{\partial \tau} (\mathcal{L} \delta \tau) \right] \end{aligned} \quad (45)$$

This gives

$$\begin{aligned} \bar{\delta} I &= \int d\tau d\xi \left\{ \left[\frac{\partial \mathcal{L}}{\partial X^\mu} - \frac{\partial}{\partial \tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \right) - \partial_a \frac{\partial \mathcal{L}}{\partial \partial_a X^\mu} \right] \delta X^\mu \right. \\ &\quad \left. + \partial_a \left(\frac{\partial \mathcal{L}}{\partial \partial_a X^\mu} \delta X^\mu + \mathcal{L} \delta \xi^a \right) + \frac{\partial}{\partial \tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \delta X^\mu + \mathcal{L} \delta \tau \right) \right\} \end{aligned} \quad (46)$$

If we assume that the equations of motion are satisfied, then from (46) we have

$$\bar{\delta} I = \int d\tau \oint d\Sigma_a (p^a_\mu \delta X^\mu + \mathcal{L} \delta \xi^a) + \int d^n \xi (p_\mu \delta X^\mu + \mathcal{L} \delta \tau) \Big|_{\tau_1}^{\tau_2} \quad (47)$$

where

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}, \quad p^a_\mu = \frac{\partial \mathcal{L}}{\partial \partial_a X^\mu} \quad (48)$$

In the above equations (46), (47) the field variation is defined at a fixed value of the arguments τ, ξ^a :

$$\delta X^\mu \equiv X'^\mu(\tau, \xi) - X^\mu(\tau, \xi) \quad (49)$$

Let us now introduce, as usually in the field theory, the total variation

$$\bar{\delta}X^\mu = X'^\mu(\tau', \xi') - X^\mu(\tau, \xi) = \delta X^\mu + \dot{X}^\mu \delta\tau + \partial_a X^\mu \delta\xi^a \quad (50)$$

Rewriting Eq.(47) in terms of $\bar{\delta}X^\mu$ we obtain

$$\bar{\delta}I = \int d\tau \oint d\Sigma_a (p^a{}_\mu \bar{\delta}X^\mu - T^a{}_b \delta\xi^b - \theta^a \delta\tau) + \int d^n\xi (p_\mu \bar{\delta}X^\mu - \mathcal{H}_a \delta\xi^a - \mathcal{H} \delta\tau) \Big|_{\tau_1}^{\tau_2} \quad (51)$$

where

$$T^a{}_b = p^a{}_\mu \partial_b X^\mu - \mathcal{L} \delta^a{}_b \quad (52)$$

$$\theta^a = p^a{}_\mu \dot{X}^\mu \quad (53)$$

$$\mathcal{H}_a = p_\mu \partial_a X^\mu \quad (54)$$

$$\mathcal{H} = p_\mu \dot{X}^\mu - \mathcal{L} \quad (55)$$

In the first term of Eq.(51) we can take the boundary at infinity and assume that the quantities $p^a{}_\mu$, $T^a{}_b$, θ^a vanish at infinity, so that the variation is simply

$$\bar{\delta}I = \int d^n\xi (p_\mu \bar{\delta}X^\mu - \mathcal{H}_a \delta\xi^a - \mathcal{H} \delta\tau) \Big|_{\tau_1}^{\tau_2} = G(\tau_2) - G(\tau_1) \quad (56)$$

The quantity

$$G(\tau) = \int d^n\xi (p_\mu \bar{\delta}X^\mu - \mathcal{H}_a \delta\xi^a - \mathcal{H} \delta\tau) \quad (57)$$

defined at a fixed value of τ is *the generator of the total variation* $\bar{\delta}$.

With the aid of the generator G we can calculate variation of an arbitrary functional $A[\tau, \xi^a, X^\mu(\xi), p_\mu(\xi)]$ according to the Poisson bracket relation

$$\delta A = \{A, G\} \quad (58)$$

The latter relation can be proven as follows. Taking $A = X^\mu$, using Eqs.(50) and assuming the validity of Eq.(58) we have

$$\delta X^\mu = \bar{\delta}X^\mu - \partial_a X^\mu \delta\xi^a - \dot{X}^\mu \delta\tau = \{X^\mu, G\} = \frac{\delta G}{\delta p_\mu} \quad (59)$$

From (57) and (59) we have (after comparing the terms and taking into account that the variations $\delta\xi^a$, $\delta\tau$ are arbitrary):

$$\partial_a X^\mu = \{X^\mu, H_a\} \quad (60)$$

$$\dot{X}^\mu = \{X^\mu, H\} \quad (61)$$

where

$$H_a = \int d^n\xi \mathcal{H}_a, \quad H = \int d^n\xi \mathcal{H} \quad (62)$$

From (54) we have that H_a is a functional of $p_\mu(\xi)$ and $\partial_a X^\mu(\xi)$. In the Hamiltonian density \mathcal{H} (Eq. (55)) we replace, using $p_\mu = \partial\mathcal{L}/\partial\dot{X}^\mu$, the velocity \dot{X}^μ by the momentum p_μ , so that $H = \int d^n\xi \mathcal{H}$ is a functional of $X^\mu(\xi)$ and $p_\mu(\xi)$.

On the other hand, if we take $A = p_\mu$ we obtain

$$\delta p_\mu = \bar{\delta} p_\mu - \partial_a p_\mu \delta \xi^a - \dot{p}_\mu \delta \tau = \{p_\mu, G\} = -\frac{\delta G}{\delta X^\mu} \quad (63)$$

where we have defined, in analogy to Eq.(50), the total variation of the momentum

$$\bar{\delta} p_\mu = p'_\mu(\tau', \xi') - p_\mu(\tau, \xi) = \delta p_\mu + \partial_a p_\mu \delta \xi^a + \dot{p}_\mu \delta \tau \quad (64)$$

From (63) we have

$$\bar{\delta} p_\mu = \int d^n \xi' \{p_\mu(\xi), p_\nu(\xi')\} \bar{\delta} X^\nu(\xi') = 0 \quad (65)$$

$$\partial_a p_\mu = \{p_\mu, H_a\} \quad (66)$$

$$\dot{p}_\mu = \{p_\mu, H\} \quad (67)$$

Using Eq.(54) we can verify that the relations (60) and (66) are satisfied identically. Using Eq.(55) we find that Eqs.(61), (67) are equivalent to the equations of motion. Thus we have verified that Eq. (58) is satisfied for the variables $X^\mu(\xi)$ and $p_\mu(\xi)$. For a generic functional we have

$$\begin{aligned} \delta A &= \int d^n \xi' \left(\frac{\delta A}{\delta X^\mu(\xi')} \delta X^\mu(\xi') + \frac{\delta A}{\delta p_\mu(\xi')} \delta p^\mu(\xi') \right) \\ &= \int d^n \xi' \left(\frac{\delta A}{\delta X^\mu(\xi')} \frac{\delta G}{\delta p_\mu(\xi')} - \frac{\delta A}{\delta p_\mu(\xi')} \frac{\delta G}{\delta X^\mu(\xi')} \right) = \{A, G\} \end{aligned} \quad (68)$$

In the last step of Eq.(68) we have replaced δX^μ and δp_μ according to the relations (59), (63) and thus obtained the Poisson bracket. This completes the proof of Eq.(58).

The total derivative of an arbitrary functional $A[\tau, \xi^a, X^\mu(\xi), p_\mu(\xi)]$ with respect to τ and ξ^a is

$$\frac{dA}{d\tau} = \frac{\partial A}{\partial \tau} + \int d^n \xi' \left(\frac{\delta A}{\delta X^\mu(\xi')} \dot{X}^\mu(\xi') + \frac{\delta A}{\delta p_\mu(\xi')} \dot{p}_\mu(\xi') \right) \quad (69)$$

$$\frac{dA}{d\xi^a} = \frac{\partial A}{\partial \xi^a} + \int d^n \xi' \left(\frac{\delta A}{\delta X^\mu(\xi')} \partial_a X^\mu(\xi') + \frac{\delta A}{\delta p_\mu(\xi')} \partial_a p_\mu(\xi') \right) \quad (70)$$

Replacing \dot{X}^μ , \dot{p}_μ , $\partial_a X^\mu$ and $\partial_a p_\mu$ by the corresponding expressions (61),(67), (60) and (66) Eqs.(69),(70) become

$$\frac{dA}{d\tau} = \frac{\partial A}{\partial \tau} + \{A, H\} \quad (71)$$

$$\frac{dA}{d\xi^a} = \frac{\partial A}{\partial \xi^a} + \{A, H_a\} \quad (72)$$

The generator (57) can be considered as the variation of the Hamilton-Jacobi functional S :

$$G(\tau) = \bar{\delta} S = \int d^n \xi p_\mu(\xi) \bar{\delta} X^\mu(\xi) - H_a \delta \xi^a - H \delta \tau \quad (73)$$

Eq.(57) or (73) can be written in a simpler form by introducing the variation

$$\bar{\delta}_\tau X^\mu = \delta X^\mu + \dot{X}^\mu \delta\tau = X'^\mu(\tau', \xi) - X^\mu(\tau, \xi) \quad (74)$$

so that the total variation defined in Eq.(50) can be expressed in terms of $\bar{\delta}_\tau X^\mu$ according to

$$\bar{\delta} X^\mu = \bar{\delta}_\tau X^\mu + \partial_a X^\mu \delta\xi^a \quad (75)$$

Then the term $H_a \delta\xi^a$ in Eq.(73) can be absorbed into the first term and we obtain

$$G(\tau) = \bar{\delta} S = \int d^n \xi p_\mu(\xi) \bar{\delta}_\tau X^\mu(\xi) - H \delta\tau = p_{\mu(\xi)} dX^{\mu(\xi)} - H d\tau \quad (76)$$

In the last step of Eq.(76) we have taken into account the convention of the summation over the discrete index μ and the integration over the continuous index ξ ; instead of $\bar{\delta}_\tau X^\mu(\xi)$ we have used the symbol $dX^{\mu(\xi)}$ which denotes the differential of coordinates $X^{\mu(\xi)}$, i.e. an infinitesimal vector in the membrane space \mathcal{M} (see Sec.2.1). The differential $dX^{\mu(\xi)}$ is taken at fixed values of the indices μ and ξ , but at any value of the evolution parameter τ (see Eq.(74)).

Now let us consider the momentum $p_{\mu(\xi)}$ as *the canonical field* defined over the families of the trajectories $X^\mu(\tau, \xi) \equiv X^{\mu(\xi)}(\tau)$ which are solutions to the equations of motion for the action (44). The vector field $p_{\mu(\xi)}$ is tangent to the trajectories $X^{\mu(\xi)}(\tau)$ and is, in general, a function of position $X^{\mu(\xi)}$ in the membrane space \mathcal{M} , i.e. a functional of $X^\mu(\xi)$,

$$p_{\mu(\xi)} = p_{\mu(\xi)}(X^{\mu(\xi)}) \equiv p_\mu(\xi)[X^\mu(\xi)] \quad (77)$$

For the canonical field the Poisson bracket (42) becomes

$$\{p_{\mu(\xi)}, p_{\nu(\xi')}\} = \partial_{\nu(\xi')} p_{\mu(\xi)} - \partial_{\mu(\xi)} p_{\nu(\xi')} = 0 \quad (78)$$

which is the expression for curl in \mathcal{M} . In addition, one may require

$$D_{\mu(\xi)} p^{\mu(\xi)} = 0 \quad (79)$$

A system of trajectories for which (78),(79) holds is called a *coherent system of trajectories* [17]. Because of (78) the integral

$$\int dS = \int_a^b (p_{\mu(\xi)} dX^{\mu(\xi)} - H d\tau) = S[\tau_b, X_b^{\mu(\xi)}; \tau_a, X_a^{\mu(\xi)}] \quad (80)$$

is independent of the path between the points a and b in \mathcal{M} . Hence, by fixing the initial point a , we find that S is a unique function(al) of position $X^{\mu(\xi)}$ in \mathcal{M} , and also a function of τ :

$$S = S(\tau, X^{\mu(\xi)}) \quad (81)$$

This is *the Hamilton-Jacobi functional*. The total differential of S is

$$dS = \frac{\partial S}{\partial \tau} d\tau + \frac{\partial S}{\partial X^{\mu(\xi)}} dX^{\mu(\xi)} = p_{\mu(\xi)} dX^{\mu(\xi)} - H d\tau \quad (82)$$

from which we have

$$\frac{\partial S}{\partial X^{\mu(\xi)}} = p_{\mu(\xi)} \quad (83)$$

$$-\frac{\partial S}{\partial \tau} = H \quad (84)$$

Eq.(83) tells us how the momentum field $p_{\mu(\xi)}$ is related to S . (By the way, in the usual notation Eq.(83) reads $\delta S/\delta X^\mu(\xi) = p_\mu(\xi)$.) Eq.(84) is the functional Hamilton-Jacobi equation. It is equivalent to the equations of motion.

So far we were discussing the general formalism for an arbitrary action of the form (44). Now we return to our particular membrane's action (20). The corresponding Hamiltonian can be straightforwardly derived from (55), (62) by using (34) and (26). We obtain

$$H = \int d^n \xi \left[\sqrt{|f|} \frac{\Lambda}{2\kappa} \left(\frac{p^\mu p_\mu}{|f|} - \kappa^2 \right) - \Lambda^a \partial_a X^\mu p_\mu \right] \quad (85)$$

The latter expression can be cast into the compact tensor notation of the membrane space \mathcal{M} (Sec. 2.1). We introduce the metric in \mathcal{M} according to Eqs.(8),(12) with $\alpha = \kappa/\Lambda$ so that the scalar product in the Hamiltonian (85) can be written as

$$\int d^n \xi \frac{\Lambda}{\sqrt{|f|\kappa}} p^\mu p_\mu = \rho^{\mu(\xi)\nu(\xi')} p_{\mu(\xi)} p_{\nu(\xi')} = \rho_{\mu(\xi)\nu(\xi')} p^{\mu(\xi)} p^{\nu(\xi')} = p_{\mu(\xi)} p^{\mu(\xi)} \quad (86)$$

where $p_{\mu(\xi)} = p_\mu(\xi)$ and $p^{\mu(\xi)} = \rho^{\mu(\xi)\nu(\xi')} p_{\nu(\xi')} = \frac{\Lambda}{\sqrt{|f|\kappa}} p^\mu(\xi)$. Eq. (85) then becomes

$$H = \frac{1}{2} (p^{\mu(\xi)} p_{\mu(\xi)} - K) - \Lambda^a \partial_a X^{\mu(\xi)} p_{\mu(\xi)} \quad (87)$$

where K is defined in Eq.(24).

Since our action, when Λ and Λ^a are independent of τ , is invariant with respect to translations of the evolution parameter $\tau \rightarrow \tau' = \tau + a$, it follows from the Noether theorem that the Hamiltonian is a constant of motion. This follows also directly from the relation (71) which, for $A = H$, gives $\dot{H} = 0$.

Using the specific Hamiltonian (85) in the Hamiltonian equation (67) we have explicitly

$$\dot{p}_\mu = -\frac{\delta H}{\delta X^\mu} = -\frac{\partial \mathcal{H}}{\partial X^\mu} + \partial_a \frac{\partial \mathcal{H}}{\partial \partial_a X^\mu} = -\kappa \partial_a (\sqrt{|f|} \mu \partial^a X_\mu + \sqrt{|f|} \frac{\Lambda^a}{\Lambda} \partial X_\mu) \quad (88)$$

where ∂X_μ and μ are given in Eqs. (26) and (27), respectively. Eq.(88) coincides with the equation of motion (25) derived directly from variation of the action (20).

Similarly, we can explicitly verify also Eq.(61):

$$\dot{X}^\mu = \frac{\delta H}{\delta p_\mu(\xi)} = \int d^n \xi' \left[\frac{\Lambda}{\sqrt{|f|\kappa}} p^\mu(\xi') - \Lambda^a \partial_a X^\mu(\xi') \right] \delta(\xi' - \xi) = \frac{\Lambda}{\sqrt{|f|\kappa}} p^\mu - \Lambda^a \partial_a X^\mu \quad (89)$$

Using (26) and (34) we see that Eq.(89) is indeed an identity.

4. The quantum theory of unconstrained membranes

4.1. The commutation relations and the Heisenberg equations of motion

Since there is no constraints among the dynamical variables $X^\mu(\xi)$ and the corresponding canonical momenta $p_\mu(\xi)$, quantization of the theory is straightforward. The classical variables become operators and the Poisson brackets are replaced by commutators, taken at equal τ . Instead of Eqs.(41),(42) we have

$$[X^\mu(\xi), p_\nu(\xi')] = i \delta^\mu_\nu \delta(\xi - \xi') \quad (90)$$

$$[X^\mu(\xi), X^\nu(\xi')] = 0, \quad [p_\mu(\xi), p_\nu(\xi')] = 0 \quad (91)$$

The quantum analog of Eq.(58) for an operator $A(\tau, \xi)$ is

$$\delta A = -i [A, G] \quad (92)$$

implying, in the case of $G(\tau)$ (Eq.(57), now considered as an operator,

$$\partial_a A = -i [A, H_a] \quad (93)$$

$$\dot{A} = -i [A, H] \quad (94)$$

Eq.(94) is the Heisenberg equation of motion for an operator A . In particular we have

$$\dot{p}_\mu(\xi) = -i [p_\mu(\xi), H] \quad (95)$$

$$\dot{X}^\mu(\xi) = -i [X^\mu(\xi), H] \quad (96)$$

We may use the representation in which the operators $X^\mu(\xi)$ are diagonal and

$$p_\mu(\xi) = -i \left(\frac{\delta}{\delta X^\mu(\xi)} + \frac{\delta F}{\delta X^\mu(\xi)} \right) \quad (97)$$

where F is a suitable functional of X^μ (see Eq.(108)). The extra term in $p_\mu(\xi)$ is introduced in order to take into account the metric (8) of the membrane space \mathcal{M} (see also Sec. 4.2) In such a representation it is straightforward to calculate the useful commutators

$$[p_\mu(\xi), \sqrt{|f|}] = -i \frac{\delta \sqrt{|f(\xi')|}}{\delta X^\mu(\xi)} = i \partial_a \left(\sqrt{|f|} \partial^a X_\mu \delta(\xi - \xi') \right) \quad (98)$$

$$[p_\mu(\xi), \partial_a X^\nu(\xi')] = -i \frac{\delta \partial_a X^\nu(\xi')}{\delta X^\mu(\xi)} = -i \delta_\mu^\nu \partial_a \delta(\xi - \xi') \quad (99)$$

For the Hamiltonian (85) the explicit form of Eqs (95)and (96) can be obtained straightforwardly by using the commutation relations (90), (91):

$$\dot{p}_\mu(\xi) = -\partial_a \left[\frac{\Lambda}{2\kappa} \partial^a X_\mu \sqrt{|f|} \left(\frac{p^2}{|f|} + \kappa^2 \right) \right] \quad (100)$$

$$\begin{aligned}
\dot{X}^\mu(\xi) &= -i \int d\xi' [X^\mu(\xi), \frac{\Lambda}{2\kappa} \sqrt{|f|} \left(\frac{p^\alpha(\xi') p_\alpha(\xi')}{|f|} - \kappa^2 \right) - \Lambda^a \partial_a X^\alpha(\xi') p_\alpha(\xi')] \\
&= \frac{\Lambda}{\sqrt{|f|} \kappa} p^\mu(\xi') - \Lambda^a \partial_a X^\mu
\end{aligned} \tag{101}$$

We recognise that the operator equations (100), (101) have the same form as the classical equations of motion (88).

4.2. The Schrödinger representation

The above relations (90)-(96), (100), (101) are valid regardless of representation. A possible representation is one in which the basic states $|X^\mu(\xi)\rangle$ have definite values $X^\mu(\xi)$ of the membrane's position operators⁵ $\hat{X}^\mu(\xi)$. An arbitrary state $|a\rangle$ can be expressed as

$$|a\rangle = \int |X^\mu(\xi)\rangle \mathcal{D}X^\mu(\xi) \langle X^\mu(\xi)|a\rangle \tag{102}$$

where the measure $\mathcal{D}X^\mu(\xi)$ is given in Eq.(15) with $\alpha = \kappa/\Lambda$.

Now we shall write the equation of motion for the wave functional $\psi \equiv \langle X^\mu(\xi)|a\rangle$. We adopt the requirement that, in the classical limit, the wave functional equation should reproduce the Hamilton-Jacobi equation (84); the supplementary equation (83) has to arise from the corresponding quantum equation. For this aim we admit that ψ evolves with the evolution parameter τ and is a functional of $X^\mu(\xi)$:

$$\psi = \psi[\tau, X^\mu(\xi)] \tag{103}$$

It is normalized according to

$$\int \mathcal{D}X \psi^* \psi = 1 \tag{104}$$

which is a straightforward extension of the corresponding relation $\int d^4x \psi^* \psi = 1$ for the unconstraint point particle in Minkowski spacetime [5]-[9]. It is important to stress again [7]-[9] that, since (104) is satisfied at any τ , the evolution operator U which brings $\psi(\tau) \rightarrow \psi(\tau') = U\psi(\tau)$ is *unitary*.

The following equations are assumed to be satisfied ($\rho = \text{Det } \rho_{\mu(\xi)\nu(\xi')}$):

$$-i\hbar \frac{1}{|\rho|^{1/4}} \partial_{\mu(\xi)} (|\rho|^{1/4} \psi) = \hat{p}_{\mu(\xi)} \psi \tag{105}$$

$$i\hbar \frac{1}{|\rho|^{1/4}} \frac{\partial (|\rho|^{1/4} \psi)}{\partial \tau} = H \psi \tag{106}$$

where⁶

$$H = \frac{1}{2} (\hat{p}^{\mu(\xi)} \hat{p}_{\mu(\xi)} - K) - \Lambda^a \partial_a X^{\mu(\xi)} \hat{p}_{\mu(\xi)} \tag{107}$$

⁵ When necessary, we use symbols with hat, in order to distinguish operators from their eigenvalues.

⁶ Using the commutator (99) we find that $\Lambda^a [p_{\mu(\xi)}, \partial_a X^{\mu(\xi)}] = \int d\xi d\xi' \Lambda^a [p_{\mu(\xi)}, \partial_a X^{\nu(\xi')}] \delta_{\nu}^{\mu} \delta(\xi - \xi') = 0$ so that the order of operators in the second term of Eq.(107) does not matter

When the metric $\rho_{\mu(\xi)\nu(\xi')}$ in \mathcal{M} explicitly depend on τ (which is the case when $\dot{\Lambda} \neq 0$) such a modified τ -derivative in Eq.(106) is required [18] in order to assure conservation of probability, as expressed by the τ -invariance of the integral (104).

The momentum operator given by

$$\hat{p}_{\mu(\xi)} = -i\hbar \left(\partial_{\mu(\xi)} + \frac{1}{2} \Gamma_{\mu(\xi)\nu(\xi')}^{\nu(\xi')} \right) \quad (108)$$

where $\frac{1}{2} \Gamma_{\mu(\xi)\nu(\xi')}^{\nu(\xi')} = |\rho|^{-1/4} \partial_{\mu(\xi)} |\rho|^{1/4}$, satisfies the commutation relations (90),(91) and is Hermitian with respect to the scalar product $\int \mathcal{D}X \psi^* p_{\mu(\xi)} \psi$ in \mathcal{M} .

The expression (107) for the Hamilton operator H is obtained from the corresponding classical expression (87) in which the quantities $X^{\mu(\xi)}$, $p_{\mu(\xi)}$ are replaced by the operators $\hat{X}^{\mu(\xi)}$, $\hat{p}_{\mu(\xi)}$. There is an ordering ambiguity in the definition of $\hat{p}^{\mu(\xi)} \hat{p}_{\mu(\xi)}$. Following the convention in a finite dimensional curved space [18], we use the identity $|\rho|^{1/4} \hat{p}_{\mu(\xi)} |\rho|^{-1/4} = -i\hbar \partial_{\mu(\xi)}$ and define

$$\begin{aligned} \hat{p}^{\mu(\xi)} \hat{p}_{\mu(\xi)} \psi &= |\rho|^{-1/2} |\rho|^{1/4} \hat{p}_{\mu(\xi)} |\rho|^{-1/4} |\rho|^{1/2} \rho^{\mu(\xi)\nu(\xi')} |\rho|^{1/4} \hat{p}_{\nu(\xi')} |\rho|^{-1/4} \psi \\ &= -|\rho|^{-1/2} \partial_{\mu(\xi)} (|\rho|^{1/2} \rho^{\mu(\xi)\nu(\xi')} \partial_{\nu(\xi')} \psi) = -D_{\mu(\xi)} D^{\mu(\xi)} \psi \end{aligned} \quad (109)$$

Let us derive the classical limit of equations (105),(106). For this purpose we write

$$\psi = A[\tau, X^\mu(\xi)] \exp \left[\frac{i}{\hbar} S[\tau, X^\mu(\xi)] \right] \quad (110)$$

with real A and S .

Assuming (110) and taking the limit $\hbar \rightarrow 0$, Eq.(105) becomes

$$\hat{p}_{\mu(\xi)} \psi = \partial_{\mu(\xi)} S \psi \quad (111)$$

If we assume that in Eq.(110) A is a slowly varying and S a quickly varying functional of $X^\mu(\xi)$ we find that $\partial_{\mu(\xi)} S$ is the expectation value of the momentum operator $\hat{p}_{\mu(\xi)}$.

Let us insert (110) into Eq.(106). Taking the limit $\hbar \rightarrow 0$, and writing separately the real and imaginary part of the equation, we obtain

$$-\frac{\partial S}{\partial \tau} = \frac{1}{2} (\partial_{\mu(\xi)} S \partial^{\mu(\xi)} S - K) - \Lambda^a \partial_a X^{\mu(\xi)} \partial_{\mu(\xi)} S \quad (112)$$

$$\frac{1}{|\rho|^{1/2}} \frac{\partial}{\partial \tau} (|\rho|^{1/2} A^2) + D_{\mu(\xi)} [A^2 (\partial^{\mu(\xi)} S - \Lambda^a \partial_a X^{\mu(\xi)})] = 0 \quad (113)$$

Eq.(112) is just the functional Hamilton-Jacobi equation (84) of the classical theory. Eq.(113) is the continuity equation, where $\psi^* \psi = A^2$ is the *probability density* and

$$A^2 (\partial^{\mu(\xi)} S - \Lambda^a \partial_a X^{\mu(\xi)}) = j^{\mu(\xi)} \quad (114)$$

is the *probability current*. While the covariant components $\partial_{\mu(\xi)} S$ (Eq.34) form a momentum vector p_μ , the contravariant components form a vector ∂X^μ (Eq.(26)):

$$\partial^{\mu(\xi)} S = \rho^{\mu(\xi)\nu(\xi')} \partial_{\nu(\xi')} S = \int d\xi' \frac{\Lambda}{\kappa \sqrt{|f|}} \eta^{\mu\nu} \delta(\xi - \xi') \frac{\delta S}{\delta X^\nu(\xi')} = \frac{\Lambda}{\kappa \sqrt{|f|}} \eta^{\mu\nu} \frac{\delta S}{\delta X^\nu(\xi)} = \partial X^\mu \quad (115)$$

where we have taken $\delta S/\delta X^\nu(\xi) = p_\nu(\xi) = \frac{\kappa\sqrt{|f|}}{\Lambda} \partial X_\nu(\xi)$ (see Eq.(34)), and raised the index by $\eta^{\mu\nu}$, so that $\partial X^\mu(\xi) = \eta^{\mu\nu} \partial X_\nu(\xi)$. So we have $\partial^{\mu(\xi)} S - \Lambda^a \partial_a X^{\mu(\xi)} = \dot{X}^{\mu(\xi)}$, and the current (114) is proportional to the velocity, as it should be.

Since Eq.(106) gives the correct classical limit, it is consistent and can be taken as the equation of motion for the wave functional ψ . We shall call (106) *the (functional) Schrödinger equation*. In general, it admits the following *continuity equation*:

$$\frac{1}{|\rho|^{1/2}} \frac{\partial}{\partial \tau} (|\rho|^{1/2} \psi^* \psi) + D_{\mu(\xi)} j^{\mu(\xi)} = 0 \quad (116)$$

where

$$\begin{aligned} j^{\mu(\xi)} &= \frac{1}{2} \psi^* (\hat{p}^{\mu(\xi)} - \Lambda^a \partial_a X^{\mu(\xi)}) \psi + \text{h.c.} \\ &= -\frac{i}{2} (\psi^* \partial^{\mu(\xi)} \psi - \psi \partial^{\mu(\xi)} \psi^*) - \Lambda^a \partial_a X^{\mu(\xi)} \psi^* \psi \end{aligned} \quad (117)$$

For exercise, we prove bellow that the probability current (117) satisfies (116). First we observe that

$$\frac{\delta \partial'_a X^{\mu(\xi')}}{\delta X^\nu(\xi)} = \partial'_a \left(\frac{\delta X^{\mu(\xi')}}{\delta X^\nu(\xi)} \right) = \delta^\mu{}_\nu \partial'_a \delta(\xi' - \xi) = -\delta^\mu{}_\nu \partial_a \delta(\xi' - \xi) \quad (118)$$

Then we calculate

$$\begin{aligned} \partial_{\nu(\xi)} (\Lambda^a \partial_a X^{\mu(\xi')} \psi^* \psi) &= \frac{\delta}{\delta X^\nu(\xi)} (\Lambda^a \partial_a X^{\mu(\xi')} \psi^* \psi) \\ &= -\Lambda^a \delta^\mu{}_\nu \partial_a \delta(\xi' - \xi) \psi^* \psi + \Lambda^a \partial_a X^{\mu(\xi')} \left(\psi \frac{\delta \psi^*}{\delta X^\nu(\xi)} + \psi^* \frac{\delta \psi}{\delta X^\nu(\xi)} \right) \end{aligned} \quad (119)$$

Multiplying (119) by $\delta^\mu{}_\nu \delta(\xi' - \xi) d\xi' d\xi$ suming over μ, ν and integrating over ξ', ξ , we obtain

$$\begin{aligned} \partial_{\mu(\xi)} (\Lambda^a \partial_a X^{\mu(\xi)} \psi^* \psi) &= -N \int d\xi' d\xi \Lambda^a \delta(\xi' - \xi) \partial_a \delta(\xi' - \xi) \psi^* \psi \\ &\quad + \int d\xi \Lambda^a \partial_a X^{\mu(\xi)} \left(\psi \frac{\delta \psi^*}{\delta X^\mu(\xi)} + \psi^* \frac{\delta \psi}{\delta X^\mu(\xi)} \right) \\ &= \Lambda^a \partial_a X^{\mu(\xi)} (\psi \partial_{\mu(\xi)} \psi^* + \psi^* \partial_{\mu(\xi)} \psi) \end{aligned} \quad (120)$$

In Eq.(120) we have taken $\delta^\mu{}_\nu \delta^\nu{}_\mu = N$ and $\int d\xi \Lambda^a \delta(\xi' - \xi) \partial_a \delta(\xi' - \xi) = 0$. Next we take into account $D_{\mu(\xi)} (\Lambda^a \partial_a X^{\mu(\xi)} \psi^* \psi) = (D_{\mu(\xi)} \partial_a X^{\mu(\xi)}) \Lambda^a \psi^* \psi + \partial_a X^{\mu(\xi)} \partial_{\mu(\xi)} (\Lambda^a \psi^* \psi)$ and

$$D_{\mu(\xi)} \partial_a X^{\mu(\xi)} = \partial_{\mu(\xi)} \partial_a X^{\mu(\xi)} + \Gamma_{\mu(\xi)\nu(\xi')}^{\mu(\xi)} \partial'_a X^{\nu(\xi')} \quad (121)$$

From (117),(120) and (121) we have

$$D_{\mu(\xi)} j^{\mu(\xi)} = -\frac{i}{2} (\psi^* D_{\mu(\xi)} \psi - \psi D_{\mu(\xi)} \psi^*)$$

$$- \Lambda^a \partial_a X^{\mu(\xi)} (\psi^* \partial_{\mu(\xi)} \psi + \psi \partial_{\mu(\xi)} \psi^* + \Gamma_{\mu(\xi)\nu(\xi')}^{\nu(\xi')} \psi^* \psi) \quad (122)$$

Using the Schrödinger equation $i|\rho|^{-1/4} \partial(|\rho|^{1/4} \psi) / \partial \tau = H \psi$ and the complex conjugate equation $-i|\rho|^{-1/4} \partial(|\rho|^{1/4} \psi^*) / \partial \tau = H^* \psi^*$ where H is given in (107) we obtain that the continuity equation (116) is indeed satisfied by the probability density $\psi^* \psi$ and the current (117). For the Ansatz (110) the current (117) becomes equal to the expression (114) as it should.

We notice that the term with $\Gamma_{\mu(\xi)\nu(\xi')}^{\nu(\xi')}$ in Eq.(122) is canceled by the same type of the term in H . The latter term in H comes from the definition (108) of the momentum operator in a (curved) membrane space \mathcal{M} , whilst the analogous term in Eq.(122) results from the covariant differentiation. The definition (108) of $\hat{p}_{\mu(\xi)}$, which is an extension of the definition introduced by DeWitt [18] for a finite dimensional curved space, is thus shown to be consistent also with the conservation of the current (117).

4.3 The stationary Schrödinger equation for a membrane

Evolution of a generic membrane's state $\psi[\tau, X^\mu(\xi)]$ is given by the τ -dependent functional Schrödinger equation (106) and the Hamiltonian (107). We are now going to consider solutions which have the form

$$\psi[\tau, X^\mu(\xi)] = e^{-iE\tau} \phi[X^\mu(\xi)] \quad (123)$$

where E is a constant. We shall call it *energy*, since it has analogous role as energy in non relativistic quantum mechanics. Considering the case $\dot{\Lambda} = 0$, $\dot{\Lambda}^a = 0$ and inserting the Ansatz (123) into Eq.(106) we obtain

$$\left(-\frac{1}{2} D^{\mu(\xi)} D_{\mu(\xi)} + i \Lambda^a \partial_a X^{\mu(\xi)} (\partial_{\mu(\xi)} + \frac{1}{2} \Gamma_{\mu(\xi)\nu(\xi')}^{\nu(\xi')}) - \frac{1}{2} K \right) \phi = E \phi \quad (124)$$

So far membrane's dimension and signature were not specified. Let us now consider *Case 2* of Sec 2.2. All the dimensions of our membrane have the same signature, and the index a of membranes' coordinates assumes the values $a = 1, 2, \dots, n = p$. Assuming a real ϕ Eq.(124) becomes

$$\left(-\frac{1}{2} D^{\mu(\xi)} D_{\mu(\xi)} - \frac{1}{2} K - E \right) \phi = 0 \quad (125)$$

$$\Lambda^a \partial_a X^{\mu(\xi)} (\partial_{\mu(\xi)} + \frac{1}{2} \Gamma_{\mu(\xi)\nu(\xi')}^{\nu(\xi')}) \phi = 0 \quad (126)$$

These are equations for a *stationary state*. They remind us of the well known p -brane equations [14].

In order to obtain from (125),(126) the conventional p -brane equations we have to assume that Eqs.(125),(126) hold for any Λ and Λ^a , which is indeed the case. Then instead of Eqs.(125),(126) in which we have the integration over ξ , we obtain the equations without the integration over ξ :

$$\left(-\frac{\Lambda}{2\kappa|f|} \eta^{\mu\nu} \frac{D^2}{DX^\mu(\xi)DX^\nu(\xi)} - \frac{\Lambda}{2} - \mathcal{E} \right) \phi = 0 \quad (127)$$

$$\partial_a X^\mu(\xi) \left(\frac{\delta}{\delta X^\mu(\xi)} + \frac{1}{2} \Gamma_{\mu(\xi)\nu(\xi')}^{\nu(\xi')} \right) \phi = 0 \quad (128)$$

The last equations are obtained from (125),(126) after writing the energy as the integral of the energy density \mathcal{E} over the membrane, $E = \int d^n \xi \sqrt{|f|} \mathcal{E}$, taking into account that $K = \int d^n \xi \sqrt{|f|} \kappa \Lambda$, and omitting the integration over ξ .

Equations (127),(128), with $\mathcal{E} = 0$, are indeed the quantum analogs of the classical p -brane constraints used in the literature [1],[14] and their solution ϕ represent states of a conventional, constrained, p -brane with the tension κ . When $\mathcal{E} \neq 0$ the preceding statement still holds, provided that $\mathcal{E}(\xi)$ is proportional to $\Lambda(\xi)$, so that the quantity $\kappa(1 - 2\mathcal{E}/\Lambda)$ is a constant, identified with the effective tension. Only the particular stationary states (as indicated above) correspond to the conventional, Dirac-Nambu-Goto p -brane states, but in general they correspond to a sort of the wiggly membranes [15, 7, 8].

4.4. Dimensional reduction of the Schrödinger equation

Let us now consider the *Case 1*. Our membrane has signature (+ - - - ...) and is actually an n -dimensional worldsheet. The index a of the worldsheet coordinates ξ^a assumes the values $a = 0, 1, 2, \dots, p$, where $p = n - 1$.

Among all possible wave functional satisfying Eqs.(106) there are also the special ones for which it holds (for an example see see Eqs.(156), (159)-(161))

$$\frac{\delta \psi}{\delta X^\mu(\xi^0, \xi^i)} = \delta(\xi^0 - \xi_\Sigma^0) (\partial_0 X^\mu \partial_0 X_\mu)^{1/2} \frac{\delta \psi}{\delta X^\mu(\xi_\Sigma^0, \xi^i)} \quad , \quad i = 1, 2, \dots, p = n - 1 \quad (129)$$

where ξ_Σ^0 is a fixed value of the time like coordinate ξ^0 . In the compact tensorial notation in membrane's space \mathcal{M} Eq.(129) reads

$$\partial_{\mu(\xi^0, \xi^i)} \phi = \delta(\xi^0 - \xi_\Sigma^0) (\partial_0 X^\mu \partial_0 X_\mu)^{1/2} \partial_{\mu(\xi_\Sigma^0, \xi^i)} \psi \quad (130)$$

Using (130) we find that dimension of the Laplace operator in \mathcal{M} is lowered:

$$\begin{aligned} D^{\mu(\xi)} D_{\mu(\xi)} \psi &= \int d^n \xi \frac{\Lambda}{\kappa \sqrt{|f|}} \eta^{\mu\nu} \frac{D^2 \psi}{DX^\mu(\xi) DX^\nu(\xi)} \\ &= \int d\xi^0 d^p \xi \frac{\Lambda}{\kappa \sqrt{|f|}} (\partial_0 X^\mu \partial_0 X_\mu)^{1/2} \eta^{\mu\nu} \delta(\xi^0 - \xi_\Sigma^0) \frac{D^2 \psi}{DX^\mu(\xi_\Sigma^0, \xi^i) DX^\nu(\xi_\Sigma^0, \xi^i)} \\ &= \int d^p \xi \frac{\Lambda}{\kappa \sqrt{|f|}} \eta^{\mu\nu} \frac{D^2 \psi}{DX^\mu(\xi_\Sigma^0, \xi^i) X^\nu(\xi_\Sigma^0, \xi^i)} = \int d^p \xi \frac{\Lambda}{\kappa \sqrt{|f|}} \eta^{\mu\nu} \frac{D^2 \psi}{DX^\mu(\xi^i) DX^\nu(\xi^i)} \end{aligned} \quad (131)$$

Here $\bar{f} \equiv \det \bar{f}_{ij}$ is the determinant of the induced metric $\bar{f}_{ij} \equiv \partial_i X^\mu \partial_j X_\mu$ on V_p , and it is related to the determinant $f \equiv \det f_{ab}$ of the induced metric $f_{ab} = \partial_a X^\mu \partial_b X_\mu$ on V_{p+1} according to $f = \bar{f} \partial_0 X^\mu \partial_0 X_\mu$ (see refs. [19], [7]-[9]).

The differential operator in the last expression of Eq.(131) (where we identified $X^{\mu(\xi_\Sigma^0, \xi^i)} \equiv X^{\mu(\xi^i)}$) acts in the space of p -dimensional membranes, though the original operator we started from acted in the space of $(p+1)$ -dimensional membranes ($n = p+1$). This comes from the fact that our special functional, satisfying (129), has vanishing functional derivative $\delta\psi/\delta X^\mu(\xi^0, \xi^i)$ for all values of ξ^0 , except for $\xi^0 = \xi_\Sigma^0$. Expression (129) has its finite dimensional analog in the relation $\partial\phi/\partial x^A = \delta_A^\mu \partial\phi/\partial x^\mu$, $A = 0, 1, 2, 3, \dots, 3+m$, $\mu = 0, 1, 2, 3$, which says that the field $\phi(x^A)$ is constant along the extra dimensions. For such a field the $(4+m)$ -dimensional Laplace expression $\eta^{AB} \frac{\partial^2 \phi}{\partial x^A \partial x^B}$ reduced to the 4-dimensional expression $\eta^{\mu\nu} \frac{\partial^2 \phi}{\partial x^\mu \partial x^\nu}$.

The above procedure can be performed not only for ξ^0 , but for any of the coordinates ξ^a ; it applies both to *Case 1* and *Case 2*.

Using (130),(131) we thus find that for such a special wave functional ψ the equation (106), which describes a state of a $(p+1)$ -dimensional membrane, reduces to the equation for a p -dimensional membrane. This an important finding. Namely, at the beginning we may assume a certain dimension of a membrane and then consider lower dimensional membranes as particular solutions to the higher dimensional equation. This means that the point particle theory (0-brane), the string theory (1-brane), and in general a p -brane theory for arbitrary p are all contained in the theory of a $(p+1)$ -brane.

4.5 A particular solution to the covariant Schrödinger equation

Let us now consider the covariant functional Schrödinger equation (106) with the Hamiltonian operator (107). The quantities Λ^a are arbitrary in principle. For simplicity we take now $\Lambda^a = 0$. Additionally we also take a τ -independent Λ , so that $\dot{\rho} = 0$. Then Eq.(106) becomes simply ($\hbar = 1$)

$$i \frac{\partial \psi}{\partial \tau} = -\frac{1}{2} (D^{\mu(\xi)} D_{\mu(\xi)} + K) \psi \quad (132)$$

The operator on the right hand side is the infinite dimensional analog of the covariant Klein-Gordon operator. Using the definition of the covariant derivative (17) and the corresponding affinity we have [9]

$$D_{\mu(\xi)} D^{\mu(\xi)} \psi = \rho^{\mu(\xi)\nu(\xi')} D_{\mu(\xi)} D_{\nu(\xi')} \psi = \rho^{\mu(\xi)\nu(\xi')} \left(\partial_{\mu(\xi)} \partial_{\nu(\xi')} \psi - \Gamma_{\mu(\xi)\nu(\xi')}^{\alpha(\xi'')} \partial_{\alpha(\xi'')} \psi \right) \quad (133)$$

The affinity is explicitly

$$\Gamma_{\mu(\xi)\nu(\xi')}^{\alpha(\xi'')} = \frac{1}{2} \rho^{\alpha(\xi'')\beta(\xi''')} \left(\rho_{\beta(\xi''')\mu(\xi),\nu(\xi')} + \rho_{\beta(\xi''')\nu(\xi'),\mu(\xi)} - \rho_{\mu(\xi)\nu(\xi'),\beta(\xi''')} \right) \quad (134)$$

where the metric is given by (8),(12) with $\alpha = \kappa/\Lambda$ and $g_{\mu\nu} = \eta_{\mu\nu}$. Using

$$\rho_{\beta(\xi'')\mu(\xi),\nu(\xi')} = \eta_{\mu\nu} \alpha(\xi) \delta(\xi - \xi'') \frac{\delta \sqrt{|f(\xi)|}}{\delta X^\nu(\xi')} = \eta_{\mu\nu} \alpha(\xi) \delta(\xi - \xi'') \sqrt{|f(\xi)|} \partial^a X_\nu(\xi) \partial_a \delta(\xi - \xi') \quad (135)$$

the equation (133) becomes

$$\begin{aligned} D_{\mu(\xi)} D^{\mu(\xi)} \psi &= \rho^{\mu(\xi)\nu(\xi')} \frac{\partial^2 \psi}{\partial X^{\mu(\xi)} \partial X^{\nu(\xi')}} - \frac{\delta(0)}{\kappa} \int d^n \xi \frac{\delta \psi}{\delta X^\mu(\xi)} \\ &\times \left[\frac{N}{2} \frac{\Lambda}{\sqrt{|f|}} \frac{1}{\sqrt{|f|}} \partial_a (\sqrt{|f(\xi)|}) \partial^a X^\mu + \left(\frac{N}{2} + 1 \right) \Lambda \partial^a X^\mu \partial_a \left(\frac{1}{\sqrt{|f(\xi)|}} \right) + \frac{\partial^a X^\mu \partial_a \Lambda}{\sqrt{|f(\xi)|}} \right] \end{aligned} \quad (136)$$

where $N = \eta^{\mu\nu} \eta_{\mu\nu}$ is dimension of spacetime. In deriving Eq.(136) we encountered the expression $\delta^2(\xi - \xi')$ which we replaced by the corresponding approximate expression $F(a, \xi - \xi') \delta(\xi - \xi')$ where $F(a, \xi - \xi')$ is any finite function, e.g. $(1/\sqrt{\pi}a) \exp[-(\xi - \xi')^2/a^2]$, which in the limit $a \rightarrow 0$ becomes $\delta(\xi - \xi')$. The latter limit was taken after performing all the integrations, and $\delta(0)$ should be considered as an abbreviation for $\lim_{a \rightarrow 0} F(a, 0)$.

The infinity $\delta(0)$ in an expression such as (136) can be regularized by taking into account the plausible assumption that a generic physical object is actually a fractal (i.e., an object with detailed structure on all scales). The objects $X^\mu(\xi)$ which we are using in our calculations are well behaved functions with definite derivatives and should be considered as approximations to the actual physical objects. This means that a description with a given $X^\mu(\xi)$ below a certain scale a has no physical meaning. In order to make a physical sense of the expression (136), $\delta(0)$ should therefore be replaced by $F(a, 0)$. Choice of the scale a is arbitrary and determines the precision of our description. (C.f., the length of a coast depends on the scale of the units which it is measured with.)

Expression (133) is analogous to the corresponding finite dimensional expression. In analogy to a finite dimensional case, a metric tensor $\rho'_{\mu(\xi)\nu(\xi')}$ obtained from $\rho_{\mu(\xi)\nu(\xi')}$ by a coordinate transformation (5) belongs to the same space \mathcal{M} and is equivalent to $\rho_{\mu(\xi)\nu(\xi')}$. Instead of a finite number of coordinate conditions which express a choice of coordinates, we have now infinite coordinate conditions. The second term in eq.(136) becomes zero if we take

$$\frac{N}{2} \frac{\Lambda}{\sqrt{|f|}} \frac{1}{\sqrt{|f|}} \partial_a \left(\sqrt{|f(\xi)|} \partial^a X_\mu \right) + \left(\left(\frac{N}{2} + 1 \right) \Lambda \partial_a \left(\frac{1}{\sqrt{|f(\xi)|}} \right) + \frac{\partial_a \Lambda}{\sqrt{|f(\xi)|}} \right) \partial^a X_\mu = 0 \quad (137)$$

and these are just possible coordinate conditions in the membrane space \mathcal{M} . Eq.(137) together with boundary conditions determines a family of functions $X^\mu(\xi)$ for which the functional ψ is defined; in the operator theory such a family is called the domain of an operator. Choice of a family of functions $X^\mu(\xi)$ is in fact choice of coordinates (a gauge) in \mathcal{M} .

If we contract Eq.(137) by $\partial_b X_\mu$ and take into account the identity (28) we find

$$\left(\frac{N}{2} + 1 \right) \Lambda \partial_a \left(\frac{1}{\sqrt{|f(\xi)|}} \right) + \frac{\partial_a \Lambda}{\sqrt{|f(\xi)|}} = 0 \quad (138)$$

From (138) and (137) we have

$$\frac{1}{\sqrt{|f(\xi)|}} \partial_a \left(\sqrt{|f(\xi)|} \partial^a X_\mu \right) = 0 \quad (139)$$

Interestingly, the gauge condition (137) in \mathcal{M} automatically implies the gauge condition (138) in V_n . The latter condition is much simplified if we take $\Lambda \neq 0$ satisfying $\partial_a \Lambda = 0$; then for $|f| \neq 0$ Eq.(138) becomes

$$\partial_a \sqrt{|f|} = 0 \quad (140)$$

which is just the gauge condition considered by Schild [3] and Eguchi [4].

In the presence of the condition (137) (which is just a gauge or coordinate condition in the function space \mathcal{M}) the functional Schrödinger equation (132) can be written in the form

$$i \frac{\partial \psi}{\partial \tau} = -\frac{1}{2} \int d^n \xi \left(\frac{\Lambda}{\sqrt{|f|} \kappa} \frac{\delta^2}{\delta X^\mu(\xi) \delta X_\mu(\xi)} + \sqrt{|f|} \Lambda \kappa \right) \psi \quad (141)$$

A particular solution to Eq.(141) can be obtained by considering the following eigenfunctions of the momentum operator

$$\hat{p}_{\mu(\xi)} \psi_p[X^{\mu(\xi)}] = p_{\mu(\xi)} \psi_p[X^{\mu(\xi)}] \quad (142)$$

with

$$\psi_p[X^{\mu(\xi)}] = \mathcal{N} \exp \left[i \int_{X_0}^X p_{\mu(\xi)} dX^{\mu(\xi)} \right] \quad (143)$$

This last expression is invariant under reparametrizations of ξ^a (Eq.(2)) and of $X^\mu(\xi^a)$ (Eq.(5)). The momentum field $p_{\mu(\xi)}$ in general functionally depends on $X^{\mu(\xi)}$ and satisfies Eqs.(78),(79). In particular $p_{\mu(\xi)}$ may be just a constant field, such that $\partial_{\nu(\xi')} p_{\mu(\xi)} = 0$. Then (143) becomes

$$\psi_p[X^{\mu(\xi)}] = \mathcal{N} \exp \left[i \int d^n \xi p_{\mu}(\xi) (X^\mu(\xi) - X_0^\mu(\xi)) \right] = \mathcal{N} \exp \left[i p_{\mu(\xi)} (X^{\mu(\xi)} - X_0^{\mu(\xi)}) \right] \quad (144)$$

The latter expression holds only in a particular parametrization (Eq.(5)) of \mathcal{M} space, but it is still invariant with respect to reparametrizations of ξ^a .

Let the τ -dependent wave functional be

$$\begin{aligned} \psi_p[\tau, X^\mu(\xi)] &= \\ \mathcal{N} \exp \left[i \int d^n \xi p_{\mu}(\xi) (X^\mu(\xi) - X_0^\mu(\xi)) \right] \exp \left[-\frac{i\tau}{2} \int \Lambda d^n \xi \left(\frac{p^\mu(\xi) p_\mu(\xi)}{\sqrt{|f|} \kappa} - \sqrt{|f|} \kappa \right) \right] \\ &\equiv \mathcal{N} \exp \left[i \int d^n \xi \mathcal{S} \right] \end{aligned} \quad (145)$$

where $f \equiv \det \partial_a X^\mu(\xi) \partial_b X_\mu(\xi)$ should be considered as a functional of $X^\mu(\xi)$ and independent of τ . From (145) we find

$$\begin{aligned} -i \frac{\delta \psi_p[\tau, X^\mu]}{\delta X^\alpha} &= -i \left(\frac{\partial \mathcal{S}}{\partial X^\alpha} - \partial_a \frac{\partial \mathcal{S}}{\partial \partial_a X^\alpha} \right) \psi_p[\tau, X^\mu] \\ &= p_\alpha \psi_p - \tau \partial_a (\sqrt{|f|} \partial^a X_\mu) \frac{\Lambda}{2\kappa} \left(\frac{p^2}{|f|} + \kappa^2 \right) \psi_p - \tau \sqrt{|f|} \partial^a X_\mu \partial_a \left[\frac{\Lambda}{2\kappa} \left(\frac{p^2}{|f|} + \kappa^2 \right) \right] \psi_p \end{aligned} \quad (146)$$

Let us now take the gauge condition (139). Additionally, let us assume

$$\partial_a \left[\frac{\Lambda}{2\kappa} \left(\frac{p^2}{|f|} + \kappa^2 \right) \right] \equiv \kappa \partial_a \mu = 0 \quad (147)$$

By inspecting the classical equations of motion (25) with $\Lambda^a = 0$ and Eq.(29) we see that Eq.(147) is satisfied when the momentum of a classical membrane does not change with τ , i.e.

$$\frac{dp_\mu}{d\tau} = 0 \quad (148)$$

Then the membrane satisfies the minimal surface equation (32) which is just our gauge condition (139) in the membrane space \mathcal{M} .⁷ When $\dot{\Lambda} = 0$, the energy $E \equiv \int \Lambda d^n \xi \left(\frac{p^\mu p_\mu}{\sqrt{|f|\kappa}} - \sqrt{|f|\kappa} \right)$ is a constant of motion. Energy conservation in the presence of Eq.(148) implies

$$\frac{d\sqrt{|f|}}{d\tau} = 0 \quad (149)$$

Our stationary state (145) is thus defined over a congruence of classical trajectories satisfying (32) and (148) which imply also (147) and (149). Eq.(146) then becomes simply

$$-i \frac{\delta \psi_p}{\delta X^\alpha} = p_\alpha \psi_p \quad (150)$$

Using (150) it is straightforward to verify that (145) is a particular solution to the Schrödinger equation (132). In Ref.[9] we found the same relation (150), but by using a different, more involved procedure.

4.6 The wave packet

From the particular solutions (145) we can compose a wave packet

$$\psi[\tau, X^\mu(\xi)] = \int \mathcal{D}p \, c[p] \, \psi_p[\tau, X^\mu(\xi)] \quad (151)$$

which is also a solution to the Schrödinger equation (132). However, since all $\psi_p[\tau, X^\mu(\xi)]$ entering (151) are supposed to belong to a restricted class of particular solutions with $p_\mu(\xi)$ which does not functionally depend on $X^\mu(\xi)$, a wave packet of the form (151) cannot represent every possible membrane's state. This is just a particular kind of wave packet; a general wave packet can be formed from a complete set of particular solutions which are not restricted to momenta $p_{\mu(\xi)}$ satisfying $\partial_{\nu(\xi')} p_{\mu(\xi)} = 0$, but allow for $p_{\mu(\xi)}$ which do depend on $X^{\mu(\xi)}$. Treatment of such a general case is beyond the scope of the present paper. Here we shall try to demonstrate some illustrative properties of the wave packet (151).

⁷The reverse is not necessarily true: the imposition of the gauge condition (139) does not imply (147),(148). The latter are additional assumptions which fix a possible congruence of trajectories (i.e. of $X^{\mu(\xi)}(\tau)$) over which the wave functional is defined.

In the definition of the invariant measure in momentum space we use the metric (12) with $\alpha = \kappa/\Lambda$:

$$\mathcal{D}p \equiv \prod_{\xi, \mu} \left(\frac{\Lambda}{\sqrt{|f|\kappa}} \right)^{1/2} dp_\mu(\xi) \quad (152)$$

Let us take⁸

$$c[p] = \mathcal{B} \exp \left[-\frac{1}{2} \int d^n \xi \frac{\Lambda}{\sqrt{|f|\kappa}} (p^\mu - p_0^\mu)^2 \sigma(\xi) \right] \quad (153)$$

where $\mathcal{B} = \lim_{\Delta\xi \rightarrow 0} \prod_{\xi, \mu} \left(\frac{\Delta\xi \sigma(\xi)}{\pi} \right)^{1/4}$ is the normalization constant, such that $\int \mathcal{D}p c^*[p] c[p] = 1$. For the normalization constant \mathcal{N} occurring in (145) we take $\mathcal{N} = \lim_{\Delta\xi \rightarrow 0} \prod_{\xi, \mu} \left(\frac{\Delta\xi}{2\pi} \right)^{1/2}$. From (151)-(153) and (145) we have

$$\begin{aligned} \psi[\tau, X(\xi)] &= \lim_{\Delta\xi \rightarrow 0} \prod_{\xi, \mu} \int \left(\frac{\Delta\xi}{2\pi} \right)^{1/2} \left(\frac{\Delta\xi \sigma(\xi)}{\pi} \right)^{1/4} \left(\frac{\Lambda}{\sqrt{|f|\kappa}} \right)^{1/2} dp_\mu(\xi) \\ &\times \exp \left[-\frac{\Delta\xi}{2} \frac{\Lambda}{\sqrt{|f|\kappa}} \left((p^\mu - p_0^\mu)^2 \sigma(\xi) - 2i \frac{\sqrt{|f|\kappa}}{\Lambda} p_\mu (X^\mu - X_0^\mu) + i\tau p_\mu p^\mu \right) \right] \\ &\times \exp \left[\frac{i\tau}{2} \int d^n \xi \sqrt{|f|\kappa} \Lambda \right] \end{aligned} \quad (154)$$

We assume no summation over μ in the exponent of the above expression and no integration (actually summation) over ξ , because these operations are now already included in the product which acts on the whole expression. Because of the factor $\left(\frac{\Delta\xi \Lambda}{\sqrt{|f|\kappa}} \right)^{1/2}$ occurring in the measure and the same factor in the exponent, the integration over p in Eq.(154) can be performed straightforwardly. The result is

$$\begin{aligned} \psi[\tau, X] &= \left[\lim_{\Delta\xi \rightarrow 0} \prod_{\xi, \mu} \left(\frac{\Delta\xi \sigma}{\pi} \right)^{1/4} \left(\frac{1}{\sigma + i\tau} \right)^{1/2} \right] \\ &\times \exp \left[\int d^n \xi \frac{\Lambda}{\sqrt{|f|\kappa}} \left(\frac{\left(i \frac{\sqrt{|f|\kappa}}{\Lambda} (X^\mu - X_0^\mu) + p_0^\mu \sigma \right)^2}{2(\sigma + i\tau)} - \frac{p_0^2 \sigma}{2} \right) \right] \exp \left[\frac{i\tau}{2} \int d^n \xi \sqrt{|f|\kappa} \Lambda \right] \end{aligned} \quad (155)$$

⁸ This can be written compactly as $c[p] = \mathcal{B} \exp \left[-\frac{1}{2} \rho_{\mu(\xi)\nu(\xi'')} (p^{\mu(\xi)} - p_0^{\mu(\xi)}) (p^{\nu(\xi')} - p_0^{\nu(\xi')}) \sigma_{(\xi')}^{(\xi'')} \right]$, where $\sigma_{(\xi')}^{(\xi'')} = \sigma(\xi') \delta(\xi', \xi'')$. Since the covariant derivative of the metric is zero, we have that $D_{\mu(\xi)} c[p] = 0$. Similarly, the measure $\mathcal{D}p = (\text{Det } \rho_{\mu(\xi)\nu(\xi'')})^{1/2} \prod_{\mu, \xi} dp_\mu(\xi)$, and the covariant derivative of the determinant is zero. Therefore $D_{\mu(\xi)} \int \mathcal{D}p c[p] \psi_p[\tau, X^\mu(\xi)] = \int \mathcal{D}p c[p] D_{\mu(\xi)} \psi_p[\tau, X^\mu(\xi)]$. This confirms that the superposition (151) is a solution if ψ_p is a solution of (106).

Eq.(155) is a generalization of the familiar Gaussian wave packet. At $\tau = 0$ Eq.(155) becomes

$$\begin{aligned} \psi[0, X] = & \left[\lim_{\Delta\xi \rightarrow 0} \Pi_{\xi, \mu} \left(\frac{\Delta\xi}{\pi\sigma} \right)^{1/4} \right] \exp \left[- \int d^n \xi \frac{\sqrt{|f|\kappa}}{\Lambda} \frac{(X^\mu(\xi) - X_0^\mu(\xi))^2}{2\sigma(\xi)} \right] \\ & \times \exp \left[i \int p_{0\mu} (X^\mu - X_0^\mu) d^n \xi \right] \end{aligned} \quad (156)$$

The probability density is given by

$$|\psi[\tau, X]|^2 = \left[\lim_{\Delta\xi \rightarrow 0} \Pi_{\xi, \mu} \left(\frac{\Delta\xi\sigma}{\pi} \right)^{1/2} \left(\frac{1}{\sigma^2 + \tau^2} \right)^{1/2} \right] \exp \left[- \int d^n \xi \frac{\sqrt{|f|\kappa}}{\Lambda} \frac{(X^\mu - X_0^\mu - \frac{\Lambda}{\sqrt{|f|\kappa}} p_0^\mu \tau)^2}{(\sigma^2 + \tau^2)/\sigma} \right] \quad (157)$$

and the normalization constant, though containing the infinitesimal $\Delta\xi$, gives precisely $\int |\psi|^2 \mathcal{D}X = 1$.

From (157) we find that the motion of the centroid membrane of our particular wave packet is determined by the equation

$$X_C^\mu(\tau, \xi) = X_0^\mu(\xi) + \frac{\Lambda}{\sqrt{|f|\kappa}} p_0^\mu(\xi) \tau \quad (158)$$

From the classical equation of motion (25) (see also (148)), (149) we indeed obtain a solution of the form (158). At this point it is interesting to observe that the classical null strings considered, within different theoretical frameworks, by Schild [3] and Roshchupkin et al. [10] also move according to the equation (158).

Function $\sigma(\xi)$ in Eqs.(153)-(157) is arbitrary; choice of $\sigma(\xi)$ determines how the wave packet is prepared. In particular, we may consider *Case 1* of Sec. 2.2 and take $\sigma(\xi)$ such that the wave packet of a $p+1$ -dimensional membrane \mathcal{V}_{p+1} is peaked around a space-like p -dimensional membrane \mathcal{V}_p . This means that the wave functional localizes \mathcal{V}_{p+1} much more sharply around \mathcal{V}_p than in other regions of spacetime. Effectively, such a wave packet describes the τ -evolution of \mathcal{V}_p (though formally it describes the τ -evolution of \mathcal{V}_{p+1}). This can be clearly seen by taking the following limiting form of the wave packet (156), such that

$$\frac{1}{\sigma(\xi)} = \frac{\delta(\xi^0 - \xi_\Sigma^0)}{\sigma(\xi^i)(\partial_0 X^\mu \partial_0 X_\mu)^{1/2}} \quad , \quad i = 1, 2, \dots, p \quad (159)$$

and choosing

$$p_{0\mu}(\xi^a) = \bar{p}_{0\mu}(\xi^i) \delta(\xi^0 - \xi_\Sigma^0) \quad (160)$$

Then the integration over δ -function gives in the exponent of Eq.(156) the expression

$$\int d^p \xi \frac{\sqrt{|f|\kappa}}{\Lambda} (X^\mu(\xi^i) - X_0^\mu(\xi^i))^2 \frac{1}{2\sigma(\xi^i)} + i \int d^p \xi \bar{p}_{0\mu} (X^\mu(\xi^i)(\xi^i) - X_0^\mu(\xi^i)) \quad (161)$$

so that Eq.(156) becomes a wave functional of a p -dimensional membrane $X^\mu(\xi^i)$. Here again \bar{f} is the determinant of the induced metric on V_p , while f is the determinant of the induced metric on V_{p+1} . One can verify that such a wave functional (156), (161) satisfies the relation (129).

The analogous considerations to those described above hold for the *Case 2* as well, so that a wave functional of a $(p - 1)$ -brane can be considered as a limiting case of a p -brane's wave functional.

5. Conclusion

We started to elaborate a theory of relativistic p -branes which is more general than the theory of conventional, constrained, p -branes. In the proposed generalized theory, p -branes are unconstrained, but among the solutions to the classical and quantum equations of motion there are also the usual, constrained, p -branes. A strong motivation for such a generalized approach is the elimination of the well known difficulties due to the presence of constraints. Since the p -brane theories are still at the stage of development and have not yet been fully confronted with observations, it makes sense to consider an enlarged set of classical and quantum p -brane states, such as e.g. proposed in the present and some previous works [7]-[9]. What we gain is a theory without constraints, still fully relativistic, which is straightforward both at the classical and the quantum level, and is not in conflict with the conventional p -brane theory.

Our approach might shed more light on some very interesting development concerning the duality [20], such as one of strings and five-branes, and the interesting interlink between p -branes of various dimensions p . In the present paper we have demonstrated how a higher dimensional p -brane equation naturally contains lower dimensional p -branes as solutions.

The highly non trivial concept of unconstrained membranes enables us to develop the elegant formulation of "point particle" dynamics in the infinite dimensional space \mathcal{M} . It is fascinating that the action, canonical and Hamilton formalism and, after quantization, the Schrödinger equation all look as nearly trivial extensions of the correspondings objects in the elegant Fock-Stueckelberg-Schwinger-DeWitt proper time formalism for a point particle in curved space. Just this "triviality", or better simplicity is a distinguished feature of our approach and we have reasons to expect that also the p -brane gauge field theory - not yet a completely solved problem - can be straightforwardly formulated along the lines indicated in the present paper.

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